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MANAGING UNCERTAINTY AND FUZZINESS TROUGH A GENERALIZED CONDITIONAL PLAUSIBILITY MODEL

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Abstract. The paper deals with a model for handling fuzziness and uncertainty simultaneously. The framework of reference is that of generalized conditional plausibility, in the sense of Dempster conditioning rule, which contains as particular cases both conditional probability and conditional possibility. Particular focus is placed on the interpretation of the interval fuzzy sets by means of this model.

Keywords: fuzziness, interval-valued fuzzy sets, uncertainty, plausibility, probability, possibility.

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1. Introduction

Starting from the seminal paper by Zadeh [44], the aim of different authors (see e.g. [6, 9, 12, 21, 22, 29, 30, 33, 36, 39, 40, 46, 47]) is to handle jointly randomness and fuzziness and, in particular, to provide a generalized Bayesian inferential procedure capable of embedding fuzziness.

An important application that has motivated this paper is related to the image reconstruction on imaging for mapping cerebral electromagnetic activity by measuring the weak magnetic field that it generates. Applications regard cognitive and functional studies to localize problematic areas of brain in order to detect neurological diseases, for example areas with spiking in epilepsy. Standard methods to address these problems are limited to the numerical methods or/and statistical ones and/or signal processing methods.

In these applications from magnetic resonance imaging of the subject a discretization of the brain is done and it is compared to a prototype of a theoretical brain, this theoretical brain is divided, given an atlas of reference in different brain regions (see e.g. [27]). It is relevant to correctly identify active brain regions in order to look for the problematic regions associated to specific features. However, it is relevant to introduce fuzzyfication of this areas to adapt the prototype of the brain region with the reconstruction of the brain of the subject.

In this kind of problems due to "imprecise" information in brain mapping, the regions (associated to some feature) are generally identified by means of an atlas, but some sites of the brain discretization could be associated to more features with a given "degree of belonging". This aspect leads to the association of each region on the atlas with a membership function on the sites. The compatibility with a given brain involves uncertainty and approximation errors. By using different atlas we can get different membership functions for some given regions over the sites.

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Then two question naturally arise:

- How to handle a family of fuzzy memberships related to the same variable?
- How to handle a family of fuzzy memberships together with uncertain on activity in these areas ?

The above questions lead to the problem to look for the mathematical connection with interval-valued fuzzy sets, that were introduced as a natural extension of fuzzy sets under different formalisms and names (see e.g. [1, 20, 34]) and with a framework able to manage together uncertainty and fuzziness.

For providing a global answer to the questions, we refer to the interpretation of membership of a fuzzy set in terms of a coherent conditional uncertainty measure, given in [5, 6] for the probabilistic framework and extended to the possibilistic framework in [4] and to the framework of theory of evidence in [11]. In these interpretations the membership of a fuzzy set is seen as a coherent conditional probability, (or possibility or plausibility) regarded as a function of the conditioning event.

These functions, from a syntactic point of view, coincide with a likelihood and so can be strictly related to that given in [18, 28, 38], which propose to reread fuzzy sets as (probabilistic) likelihood function.

Nevertheless, since usually the likelihood function is considered strictly related to the data of an experiment, to regard the membership as a coherent assessment of a conditional uncertainty measure, permits both to consider this assessment as a measure of a degree of belief of one or more subjects and to refer to different uncertainty frameworks.

In this paper we focus on the extension of this interpretations to interval-valued fuzzy sets, so that the extremes of the intervals are simply seen as two coherent conditional measures (precisely conditional probability, conditional possibility or conditional plausibility). The two membership functions can be justified in the field of managing uncertainty under partial knowledge. This interpretation implies that interval-valued fuzzy sets could represent a family of (conditional) probabilities arising when the information is partial and so it is closed to de Finetti theory of coherent conditional probabilities [14] and its connections with Walley theory of imprecise probabilities could come out.

In this paper we first introduce the fundamental concepts related to the formalization of the interpretation of fuzzy sets in the three different uncertainty frameworks, highlighting the similarities and differences.

The first result we highlight is that, when we limit ourselves to a single fuzzy subset, any assignment $\varphi(E|.)$ between 0 and 1 is, in fact, a coherent conditional plausibility $Pl(E|.)$, but also a coherent conditional probability $P(E|.)$, and a coherent conditional possibility $\Pi(E|.)$.

Then, one wonders what is the best framework for this representation that uses conditional uncertainty measures. If we limit ourselves to the syntactic point of view up to this point we have no reason to choose one or the other paradigm. However, if the values in the range of the variable of reference are affected by uncertainty, it will be the uncertainty measure to guide the choice of the framework.

Now, if, as usual, the initial data are related to several fuzzy subsets and to a measure of uncertainty on the range of the variable, it would be necessary to check coherence (consistency), with respect to the chosen uncertainty framework. Note that the two assignments $\varphi(E_i|.)$ and $\varphi(.)$ can be separately coherent, but globally not coherent. Nevertheless, since the involved events E_i naturally satisfy a form of logical independence, this guarantees the overall consistency of the initial assignment.

The second result is on the extension of the initial assessment to events that are union, intersection and complementation of the initial E_i. The computation of the membership of these new events is necessarily subject to the rules of the uncertainty measure of reference. For them we study the intervals of coherence and the connections with the t-norms and t-conorms used in the fuzzy set theory.

In this way we can give a syntactical motivation for choosing some particular t-norms and t-conorms (such as the minimum and maximum), instead of others (such as Lukasiewicz t-norm and t-conorm). After this comparative study we are able to easily introduce the interpretation of IVF's in these frameworks and provide a propagation of these values for the memberships on other events such as fuzzy events. A simple example show our proposal from both a syntactic and a semantic point of view.

2. Uncertainty framework

We briefly recall some definitions and results related to coherent conditional uncertainty measure (focusing on plausibility and in two special members of this class, i.e. probability and possibility).

We recall that a *plausibility function* Pl [15, 37] on a Boolean algebra $\mathcal{A}$ is a function such that $Pl(\emptyset) = 0$, $Pl(S) = 1$ and is n-alternating for every $n \geq 2$, i.e., for every for every finite family, $A_1, \ldots, A_n \in \mathcal{A}$,

$$Pl\left(\bigwedge_{i=1}^{n} A_i\right) \leq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Pl\left(\bigvee_{i \in I} A_i\right).$$

The previous property implies the monotonicity of Pl with respect to set inclusion $\subseteq$, hence plausibility functions are particular *normalized capacities* [16]. The *dual* function Bel defined, for every $A \in \mathcal{A}$, as $Bel(A) = 1 - Pl(A^c)$, is called *belief function.*

A plausibility function Pl on $\mathcal{A}$ is completely singled out by its *Möbius inverse* called *basic probability assignment* [37], defined for every $A \in \mathcal{A}$ as

$$m(A) = \sum_{B \cap A \neq \emptyset} (-1)^{|A \setminus B|} Pl(B).$$

Such a function $m : \mathcal{A} \to [0, 1]$ satisfies the following conditions: $m(\emptyset) = 0$, $\sum_{A \in \mathcal{A}} m(A) = 1$, and, for every $A \in \mathcal{A}$,

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \quad \text{and} \quad Bel(A) = \sum_{B \subseteq A} m(B).$$

In the literature there are many definitions of conditioning for plausibility and belief functions, we recall the following axiomatic definition (see [9, 10]):

Definition 2.1. *Let $\mathcal{A}$ be a Boolean algebra and $\mathcal{H} \subseteq \mathcal{A} \setminus \{\emptyset\}$ an additive class (i.e., a set of events closed under finite unions). A function $Pl : \mathcal{A} \times \mathcal{H} \to [0, 1]$ is a conditional plausibility function if (and only if) it satisfies the following conditions*

(i): $Pl(E|H) = Pl(E \wedge H|H)$, for every $E \in \mathcal{A}$ and $H \in \mathcal{H}$;

(ii): $Pl(\cdot|H)$ is a plausibility function on $\mathcal{A}$, for every $H \in \mathcal{H}$;

(iii): $Pl(E \wedge F|H) = Pl(E|H) \cdot Pl(F|E \wedge H)$, for every $E \wedge H, H \in \mathcal{H}$ and $E, F \in \mathcal{A}$.

Moreover, given a conditional plausibility function, the dual conditional belief function $Bel(\cdot|\cdot)$ is defined for every event $E|H \in \wp(S) \times \mathcal{H}$ as

$$Bel(E|H) = 1 - Pl(E^c|H).$$

An easy consequence of Definition 2.1. is a weak form of disintegration formula [11] for the plausibility of any conditional event $E|H$ with respect to a partition $H_1, ..., H_N$ of H

$$Pl(E|H) \leq \sum_{k=1}^{N} Pl(H_k|H)Pl(E|H_k). \tag{1}$$

Other different definitions of conditioning are present in the literature: the most interesting and famous is that due to Jaffray and Walley (see [25, 42]) defined by the following equation :

$$Pl(F|H) = \frac{Pl(F \wedge H)}{Pl(F \wedge H) + Bel(F^c \wedge H)} \tag{2}$$

and obtained as upper envelope of particular classes of conditional probabilities.

We notice that conditional plausibility defined by equation (2) does not satisfy axiom *(iii)* of Definition 2.1. and then, in particular, it does not satisfy (1).

Finally we point out that the class of conditional plausibilities (defined in Definition 2.1.) contains in particular important classes: conditional probabilities, as introduced in [13, 17, 26], and T-conditional possibilities ([8]), with the t-norm T equal to the usual product (in symbols T_P-conditional possibility).

As it follows from the results in [11], every conditional plausibility function $Pl(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$ is completely determined by a linearly ordered class of plausibility functions on $\mathcal{A}$ with disjoint sets of focal elements, which is called *agreeing class* of plausibility functions.

As proved in [11], when $\mathcal{A}$ is finite, every conditional plausibility function $Pl(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$ is completely determined by a linearly ordered class of plausibility functions on $\mathcal{A}$ with disjoint sets of focal elements, which is called *minimal agreeing class* of plausibility functions. In general, if $\mathcal{H} \subset \mathcal{A} \setminus \{\emptyset\}$ such a class is not unique, but uniqueness is obtained in case $\mathcal{H} = \mathcal{A} \setminus \{\emptyset\}$. Among the agreeing classes giving rise to a $Pl(\cdot|\cdot)$ on $A \times \mathcal{H}$ there is a unique agreeing class $\{Pl_0, \ldots, Pl_k\}$ of plausibility functions on $\mathcal{A}$, called *minimal agreeing class*, such that

- $Pl_0(\cdot) = Pl(\cdot|H_0^0)$ with $H_0^0 = \bigcup_{H \in \mathcal{H}} H$;
- for $\alpha > 0$, $Pl_\alpha(\cdot) = Pl(\cdot|H_0^\alpha)$
  with $H_0^\alpha = \{H \in \mathcal{H} : Pl_\beta(H) = 0, \beta = 0, \ldots, \alpha - 1\} \neq \emptyset$.

The class $\{Pl_0, \ldots, Pl_k\}$ is such that for every $H \in \mathcal{H}$ there is $\alpha \in \{0, \ldots, k\}$ such that $Pl_\alpha(H) > 0$. Moreover, $\{Pl_0, \ldots, Pl_k\}$ agrees with the conditional plausibility $Pl(\cdot|\cdot)$ on $\mathcal{A} \times \mathcal{H}$ in the sense that, for every $E|H \in \mathcal{A} \times \mathcal{H}$, denoting with α_H the minimum index in $\{0, \ldots, k\}$ such that $Pl_{\alpha_H}(H) > 0$, it holds that

$$Pl(E|H) = \frac{Pl_{\alpha_H}(E \cap H)}{Pl_{\alpha_H}(H)}.$$

We now introduce the concepts of coherent conditional plausibility and coherent extension.

Definition 2.2. A real function defined on an arbitrary set of conditional events $\mathcal{G} = \{E_i|H_i\}_{i \in I}$ is a *coherent conditional plausibility assessment* if and only if it is the restriction of a *conditional plausibility* $Pl' : \mathcal{B} \times \mathcal{H} \to [0, 1]$, where $\mathcal{H}$ is the additive set spanned by $\{H_i\}_{i \in I}$ and $\mathcal{A}$ the algebra spanned by $\{H_i, E_i\}_{i \in I}$.

The main characteristic of coherent assessments on an arbitrary set of (conditional) events is their extensibility to any superset maintaining coherence ([13, 17, 26] for probability, [9] for possibilities, [11] for plausibilities).

By using the representation of a conditional plausibility trough agreeing classes, introduced before, it is possible to prove the following Theorem 2.1 characterizing a coherent conditional plausibility assessment ([11]). It extends those proved in [5, 6] and in [8] for probabilities and possibilities, respectively.

Theorem 2.1. *Let $\mathcal{G}$ be an arbitrary set of conditional events. For an assessment $\sigma : \mathcal{G}$ to [0,1] the following statements are equivalent:*

j) σ is a coherent conditional plausibility;

jj) for every finite set $\mathcal{F} \subseteq \mathcal{G}$ there exists a minimal agreeing class $\{Pl_0^{\mathcal{F}}, \ldots, Pl_k^{\mathcal{F}}\}$;

jjj) for every finite set $\mathcal{F} \subseteq \mathcal{G}$ the following systems $(S_{\mathcal{F}}^{\alpha})$, with $\alpha = 0, 1, 2, ..., k \leq n$, admit a solution $\mathbf{X}^{\alpha} = (\mathbf{x}_1^{\alpha}, ..., \mathbf{x}_{j\alpha}^{\alpha})$ with $\mathbf{x}_j^{\alpha} = m_{\alpha}(H_j)$ $(j = 1, ..., j_{\alpha})$:

$$(S_{\mathcal{F}}^{\alpha}) = \begin{cases} \sum\limits_{H_k \wedge F_i \neq \emptyset} x_k^{\alpha} \cdot Pl(E_i|F_i) = \sum\limits_{H_k \wedge E_i \wedge F_i \neq \emptyset} x_k^{\alpha}, & \forall F_i \subseteq H_0^{\alpha} \\ \sum\limits_{H_k \in H_0^{\alpha}} x_k^{\alpha} = 1 & \\ x_k^{\alpha} \geq 0, & \forall H_k \subseteq H_0^{\alpha} \end{cases}$$

where H_0^{α} is the greatest element of $\mathcal{K}$ such that $\sum\limits_{H_i \wedge H_0^{\alpha} \neq \emptyset} m_{(\alpha-1)}(H_i) = 0.$

Remark 2.1. *The previous Theorem 2.1 puts in evidence that to prove coherence of an assessment, it is sufficient to prove coherence in any finite subset. This is due to the fact that no continuity condition is required.*

Definition 2.3. *Given an event E and a partition L, a likelihood function is an assessment on $\{E|H_i : H_i \in L\}$ (that is a function $f : \{E\} \times L \to [0,1]$) satisfying (only) the following trivial condition:*

(L1) $f(E|H_i) = 0$ if $E \wedge H_i = \emptyset$ and $f(E|H_i) = 1$ if $H_i \subseteq E$.

The following Theorems 2.2 and 2.3 are the main results for representing a membership as a likelihood function in the plausibilistic, probabilistic and possibilistic frameworks.

Theorem 2.2. *Let $L = \{H_j\}_{j \in J}$ be any partition of Ω and let E be any event. For every function $f : \{E\} \times L \to [0,1]$ satisfying the condition (L1) the following statements hold:*

a) *f is a coherent conditional probability;*

b) *f is a coherent T-conditional possibility (for every continuous t-norm T);*

c) *f is a coherent conditional plausibility.*

Proof. Due to characterization of coherent assessments of conditional plausibility (the same for conditional probability [5] and T-possibility [9]) it is sufficient to prove that the statements hold for every finite partition. The proof of that follows by the obvious solvability of the systems in condition jjj) when the events H_i are a partition. □

The previous result points out that "syntactically" a probabilistic likelihood function is indistinguishable from a possibilistic likelihood function or a plausibilistic likelihood function, i.e., any function f satisfying the minimal requirement of consistence *(L1)* can be extended either as a probabilistic strategy or as a possibilistic strategy or as a plausibility strategy. Obviously, the extensions are syntactically different, so a criterion for choosing the framework must be determined. This criterion could be guided from semantic motivations or related to syntactically reasons. In the last case the choice could be ruled by the "prior" information. The following Theorem 2.3 essentially proved in [23], studies coherence of a global assessment containing a coherent assessment on a partition $L = \{H_j\}_{j \in J}$ of Ω and a finite number of likelihood functions $f_i = f(E_i|H_j)$ $(i = 1, ..., m)$.

Definition 2.4. *Let us consider any partition $L = \{H_i\}_{i \in I}$ of Ω and with two Boolean algebras $\mathcal{A}_L$ and $\mathcal{A}$ where $\mathcal{A}_L = \langle L \rangle$ is the algebra generated by L and $\mathcal{A}$ a super-algebra of $\mathcal{A}_L$.*

A *plausibilistic [probabilistic] [possibilistic] strategy* is a map $\sigma : \mathcal{A} \times L \to [0, 1]$ satisfying the following conditions for every $H_i \in L$:

(S1) $\sigma(E|H_i) = 0$ if $E \wedge H_i = \emptyset$ and $\sigma(E|H_i) = 1$ if $E \wedge H_i = H_i$, for every $E \in \mathcal{A}$;

(S2) $\sigma(\cdot|H_i)$ is a plausibility [a finitely additive probability] [a finitely maxitive possibility] on $\mathcal{A}$.

Now we recall the notion of *almost logical independence* that will have a relevant role in the following, as shown by the next Theorem 2.3.

Definition 2.5. *Let us consider any partition $L = \{H_i\}_{i \in I}$ of Ω, a set of events $\{E_i : i \in I\}$* are *almost logical independent with respect to L* if, denoting with E'_i either E_i or E^c_i, the following conditions hold:

(i) the events E_i $(i = 1, ..., m)$ are logically independent, i.e., $\bigwedge_{i \in I} E'_i \neq \emptyset$;

(ii) for every $H \in L$, $\bigwedge_{i \in I} E'_i \wedge H = \emptyset \Longrightarrow E'_i \wedge H = \emptyset$ for some $i \in I$.

Theorem 2.3. *Let L be a partition of Ω, consider a family of likelihood function f_i related to events E_i almost logically independent with respect to L and let us consider, on the algebra $\mathcal{A}_L$ generated by L, a probability P, a possibility Π, and a plausibility Pl. If the events E_i are almost logical independent with respect to L, the following statements hold:*

a) the global assessment $\{f_i, P\}$ is a coherent conditional probability;

b) the global assessment $\{f_i, \Pi\}$ is a coherent T-conditional possibility (for every continuous t-norm T);

c) the global assessment $\{f_i, Pl\}$ is a coherent conditional plausibility.

3. Interpretation of fuzzy sets trough conditional uncertainty measures

We recall the main steps to construct the model of fuzzy sets theory by referring to coherent conditional uncertainty measures.

Let X be a (possibly not numerical) variable, with range $\mathcal{C}_X$, and, for any $x \in \mathcal{C}_X$, let us indicate by x the event $\{X = x\}$.

For any *property* γ related to the variable X, let us consider the Boolean event:

$$E_\gamma = \text{“}\textit{You claim that } X \textit{ has property } \gamma\text{”},$$

where “You ” denotes any real (or fictitious) person.

First of all we remark that, if $\{\gamma_i\}_{i=1,..,m}$ is a set of properties related to the same variable X, then the Boolean events E_{γ_i} $(i = 1, ..., m)$ are *almost logical independent with respect to $\mathcal{C}_X$*.

Remark 3.1. *We notice that the almost logical independence w.r.t. $\mathcal{C}_X$ is guaranteed also when for some i and j one has $\gamma_i = \neg\gamma_j$ or when γ_i is a superlative or a diminutive property of γ_j. Indeed, You can claim both “X has the property γ_i” and “X has the property $\neg\gamma_j$”, or claim only one of them or finally claim neither of them. A similar reasoning can be made when γ_i is the superlative or a diminutive property of γ_j.*

Referring now to the state of information of You, let us consider the most appropriate model to deal with uncertainty in the reference context containing events of kind E_γ. Let $\varphi(\cdot|\cdot)$ be the conditional measure related to the chosen model.

Then, an assessment $\{\varphi(E_\gamma|x)\}_{x \in \mathcal{C}_X}$ provides the measures of *how much You believe in E_γ, when X assumes the different values in its range.*

The conditional uncertainty measures taken into account in this paper are conditional plausibilities and the two main subclasses of them, i.e. conditional probabilities and conditional possibilities.

By Theorem 2.2, it is clear the total freedom in assessing the function $\varphi(E_\gamma|\cdot)$, in each of the above frameworks for dealing with uncertainty. In fact any such assessment is only required to satisfy the trivial consistency condition $(L1)$ (and assuming any value in $[0,1]$ otherwise) can be regarded as a coherent conditional probability, a coherent conditional possibility or a coherent conditional plausibility.

Then $\varphi(E_\gamma|\cdot)$ comes out to be a natural interpretation of the membership function $\mu_\gamma(\cdot)$ and so the next definition is syntactically unexceptionable and has a semantic value for all those situations where the assignment of membership is naturally subjective.

Definition 3.1. *Let X be a variable with range $\mathcal{C}_X$, γ a property related to X and φ a coherent conditional plausibility (or, more specifically, probability or possibility) on a set of conditional events containing $E_\gamma|x$, for every $x \in \mathcal{C}_X$. A* **fuzzy subset** *E^*_γ of $\mathcal{C}_X$ is any pair*

$$E^*_\gamma = (E_\gamma, \mu_\gamma),$$

with $\mu_\gamma(x) = \varphi(E_\gamma|x)$ for every $x \in \mathcal{C}_X$.

Obviously under this interpretation of fuzzy sets it is necessary to study the role of the conditional uncertainty measures when one considers more than one fuzzy set.

Indeed a good model must guarantee the possibility of maintaining the same freedom in assessing memberships related to different properties and assuring the possibility to use the usual logical paradigm for combining fuzzy sets.

Theorem 2.2 assures that, for every finite family of fuzzy sets $\{E_{\gamma_i}\}$, almost logically independent with respect to $\mathcal{C}_X$, every assessment $\varphi(E_{\gamma_i}|x)$ is a coherent conditional plausibility (a coherent conditional possibility and a coherent conditional probability).

Now, referring to the considerations made in Remark 3.1., it is clear that the elements which usually form the initial data of a problem, are almost logically independent and so the assessment of memberships is free.

Now the problem is to combine different fuzzy sets referred to the same variable X, that is to introduce operations of complementation, union and intersection, in a way that coherence is maintained. Obviously, these operations depend on the chosen framework of reference.

By following [5, 6] for the probabilistic interpretation and [4] for the possibilistic interpretation, the operation of complementation of a fuzzy set E^*_γ and those of union and intersection between two fuzzy sets E^*_γ and E^*_δ, can be directly obtained by using the rules of coherent conditional plausibility [probability or possibility] and the logical independence between E_γ and E_δ with respect to the partition generated by the relevant variable.

Let us denote by $\gamma \vee \delta$ and $\gamma \wedge \delta$, respectively, the properties "γ or δ", "γ and δ".

Note that the symbols $\wedge$ and $\vee$ do not indicate Boolean operations, since γ and δ are not Boolean objects.

Let us define:

$$E_{\gamma\vee\delta} = E_\gamma \vee E_\delta, \quad E_{\gamma\wedge\delta} = E_\gamma \wedge E_\delta. \tag{3}$$

Let us consider now the relevant fuzzy sets $E^*_\gamma = (E_\gamma, \mu_\gamma(x))$ and $E^*_\delta = (E_\delta, \mu_\delta(x))$ and define

$$E^*_\gamma \cup E^*_\delta = E^*_{\gamma\vee\delta} = (E_{\gamma\vee\delta},\, \mu_{\gamma\vee\delta}(x)), \qquad E^*_\gamma \cap E^*_\delta = E^*_{\gamma\wedge\delta} = (E_{\gamma\wedge\delta},\, \mu_{\gamma\wedge\delta}(x)).$$

The rules of the different considered uncertainty measures induce the following constraints:

- if μ is a coherent conditional probability one has:

$$\mu_{\gamma\vee\delta}(x) = \mu_\gamma(x) + \mu_\delta(x) - \mu_{\gamma\wedge\delta}(x). \tag{4}$$

This implies that we need to refer only to t-norms and their dual t-conorms in the class of Frank [23], i.e., those defined for $x, y \in [0, 1]$ and $\lambda \in [0, +\infty]$, as

$$T^F_\lambda(x, y) = \log_\lambda \left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right). \tag{5}$$

Distinguished instances of previous class are:
– the minimum $T_M(x, y) = T^F_0(x, y) = \min\{x, y\}$,
– the algebraic product $T_P(x, y) = T^F_1(x, y) = x \cdot y$,
– the Lukasiewicz t-norm $T_L(x, y) = T^F_{+\infty}(x, y) = \max\{x + y - 1, 0\}$.
The relevant dual t-conorms are listed
– $S_M(x, y) = S^F_0(x, y) = \max\{x, y\}$,
– $S_P(x, y) = S^F_1(x, y) = x + y - x \cdot y$,
– $S_L(x, y) = S^F_{+\infty}(x, y) = \min\{x + y, 1\}$.

Moreover, by Fréchet bounds, one has that, for any given $x \in \mathcal{C}_X$ the assessment $\mu_\gamma(x)$, $\mu_\delta(x)$ and $\mu_{\gamma\wedge\delta}(x)$ is coherent if and only if it holds

$$T_L(\mu_\gamma(x), \mu_\delta(x)) \leq \mu_{\gamma\wedge\delta}(x) \leq T_M(\mu_\gamma(x), \mu_\delta(x)) \tag{6}$$

and so, by equation (4),

$$S_M(\mu_\gamma(x), \mu_\delta(x)) \leq \mu_{\gamma\vee\delta}(x) \leq S_L(\mu_\gamma(x), \mu_\delta(x)). \tag{7}$$

- If μ is a coherent conditional possibility, as shown in [4] one has:
for any given $x \in \mathcal{C}_X$ the assessment $\mu_\gamma(x)$, $\mu_\delta(x)$, $\mu_{\gamma\wedge\delta}(x)$ and $\mu_{\gamma\vee\delta}(x)$ is coherent if and only if

$$\mu_{\gamma\vee\delta}(x) = S_M(\mu_\gamma(x), \mu_\delta(x)) \tag{8}$$

and

$$0 \leq \mu_{\gamma\wedge\delta}(x) \leq T_M(\mu_\gamma(x), \mu_\delta(x)) \tag{9}$$

- If μ is a coherent conditional plausibility, as shown in [4, 11] one has:
for any given $x \in \mathcal{C}_X$ the assessment $\mu_\gamma(x)$, $\mu_\delta(x)$, $\mu_{\gamma\wedge\delta}(x)$ and $\mu_{\gamma\vee\delta}(x)$ is coherent if and only if

$$0 \leq \mu_{\gamma\wedge\delta}(x) \leq T_M(\mu_\gamma(x), \mu_\delta(x)) \tag{10}$$

and

$$S_M(\mu_\gamma, \mu_\delta) \leq \mu_{\gamma\vee\delta}(x) \leq \min(\mu_\gamma(x) + \mu_\delta(x)) - \mu_{\gamma\wedge\delta}(x), 1). \tag{11}$$

Remark 3.2. *We notice that in the probabilistic interpretation, fixed the value for the membership function of the fuzzy intersection, the value for the membership function of the fuzzy union is uniquely determined* [6]*, by the equation* (4)*.*

On the contrary, in the possibilistic interpretation [4], independently of the value of $\mu_{\gamma\wedge\delta}(x) = \Pi(E_\gamma \wedge E_\delta|x)$, for the fuzzy union we get the unique value obtained by equation *(8)*.

In the case of plausibility framework the value of $\mu_{\gamma\vee\delta}(x)$ is not univocally determined, but it must satisfy the constrain given by equation *(11)*.

Obviously, since probabilities and possibilities are particular plausibilities, any pair of t-norm and t-conorm on Frank's class and any pair $(\max, T)$, with T any t-norm, can be used to compute the union and intersection of fuzzy sets.

On the contrary no pair in other famous classes (Hamacher class [24], Yager class [41], Dubois and Prade class [19]) in these classes there exist $(x', y'), (x'', y'') \in [0,1]^2$ such that

$$T(x', y') < x' + y' - S(x', y'); \qquad T(x'', y'') > x'' + y'' - S(x'', y''),$$

so that the extension of the assessment

$$\mu_\gamma(x) = Pl(E_\gamma|x), \ \ \mu_\delta(x) = Pl(E_\delta|x)$$

to $\mu_{\gamma\wedge\delta}(x)$ and $\mu_{\gamma\vee\delta}(x)$ trough (T, S) could re-sult either not two-alternating and not two-monotone and so the extension cannot be a coherent conditional plausibility. At the same time it cannot be a coherent conditional belief function.

Inequalities (9) and (11) *emphasize a first difference with the probabilistic framework (see (7)), in fact the upper bounds in the two frameworks coincide while the lower bounds differ, in fact under a probability P the lower bound is not 0, but coincides with Fréchet-Hoeffding lower bound, that is determined by the Lukasiewcz t-norm T_L, so*

$$max\{0, P(E_\gamma|x) + P(E_\delta|x) - 1\} \leq v \leq \min\{P(E_\gamma|x), P(E_\delta|x)\}.$$

All what discussed above only refers to two fuzzy sets, so it is necessary to consider the problem related to any family of fuzzy sets $\{E^*_{\gamma_1}, ..., E^*_{\gamma_n}\}$ and study if it possible to compute all the intersections among the relevant fuzzy sets, by using the same t-norm.

The answer is not unique and depends on the t-norm we use.

The following result, essentially proved in [11], shows that it is possible to compute by T_M the intersections of the elements of any finite family of fuzzy sets $\{E^*_{\gamma_i}\}$, maintaining coherence. We report the proof for completeness.

Theorem 3.1. *Let $\{E^*_{\gamma_i}\}_{i\in I}$ be a finite family of fuzzy sets related to a variable X, with $\{E_{\gamma_i}\}_{i\in I}$ almost logical independent with respect to $\mathcal{C}_X$, let*

$$E^*_{\wedge_{i\in J}\gamma_i} = \big(E_{\wedge_{i\in J}\gamma_i}, T_M(\{\mu_{\gamma_i}(x) \ with \ x \in \mathcal{C}_X \ : \ i \in J\})\big)$$

be the fuzzy sets (with $J \subseteq I$), then for the extension

$$\{\mu_{\wedge_J \gamma_i}(x) \ with \ x \in \mathcal{C}_X \ : \ i \in J\} \qquad for \ J \subseteq I$$

with

$$\mu_{\bigwedge_J \gamma_i}(x) = \varphi(\wedge_{i\in J} E_{\gamma_i}|x) = T_M\big(\mu_{\gamma_i}(x)\big) \ for \ any \ x \in \mathcal{C}_X$$

of the assessment

$$\{\mu_{\gamma_i}(x) \ with \ x \in \mathcal{C}_X \ : \ i \in J\}$$

the following statements hold:

- *the assessment is a coherent conditional plausibility*
- *the assessment is a coherent conditional probability*
- *the assessment is a coherent conditional possibility.*

Proof. From theorem 2.1 and Remark 2.1. it is sufficient to prove coherence for any finite subset of the family. Then, in this case, we need to prove the result for every finite family $\{E_{\gamma_j}|A_x\}_{x\in\mathcal{F}}$ with $\mathcal{F}$ any finite subset of $\mathcal{C}_X$.

Since the membership functions $\mu_{\gamma_j}(\cdot)$ is a coherent conditional probability (see Theorem 2.3), there is a coherent extension on $\bigwedge_{j\in J} E_{\gamma_j}|A_x$ for any $J \subseteq I$ and $x \in \mathcal{F}$.

For a given $x \in \mathcal{F}$, assume without loss of generality that $\mu_{\gamma_i}(x)) \leq \mu_{\gamma_{i+1}}(x)$ for $i = 1, 2$.

We can define for any $J_1 \subseteq I$ with $1 \in J_1$

$$f_x(\bigwedge_{i \in I} E_{\gamma_i}) = \mu_{\gamma_1}(x).$$

Moreover for a set $J \subseteq I$ with $1 \notin J$, let $r = \min\{i : i \in I \bigcap J\}$ and $s = \max\{i : i \in I \bigcap J\}$ $(r < s)$

$$f_x(\bigwedge_{j \geq r} E_{\varphi_j} \bigwedge_{i<r} E^c_{\gamma_i}) = \mu_{\gamma_r}(x) - \mu_{\gamma_{r-1}}(x)$$

and 0 on the other atoms. Any f_x is a probability, so a probability P on the algebra generated by $\{E_{\gamma_i}, A_x : i \in I, x \in \mathcal{F}\}$ can be defined as

$$P(B) = \sum_{A_x \wedge B \neq \emptyset} \frac{1}{n} f_x(B \wedge A_x)$$

(with n the cardinality of $\mathcal{F}$) and it gives rise to a strictly positive probability and it generates a conditional probability that is an extension of $\{\mu_{\gamma_i}\}_I$.

Then, the assignment f is a coherent conditional probability and then a coherent conditional plausibility.

Furthermore, the above assignment f_x, for a given $x \in \mathcal{F}$, is a possibilistic distribution and so a possibility Π on the algebra generated by $\{E_{\varphi_i}, A_x : i \in I, x \in \mathcal{F}\}$ can be defined as

$$\Pi(B) = \max_{A_x \wedge B \neq \emptyset} \frac{1}{n} f_x(B \wedge A_x)$$

(with n the cardinality of $\mathcal{F}$) and it gives rise to a strictly positive possibility and it generates a P-conditional possibility that is an extension of $\{\mu_{\varphi_i}\}_I$. □

An analogous result can be proved by considering the t-norm T_P.

Theorem 3.2. *Let $\{E^*_{\gamma_i}\}_{i \in I}$ be a finite family of fuzzy sets related to a variable X, with $\{E_{\gamma_i}\}_{i \in I}$ almost logical independent with respect to $\mathcal{C}_X$, let*

$$E^*_{\wedge_{i \in J}\gamma_i} = (E_{\wedge_{i \in J}\gamma_i}, T_P(\{\mu_{\gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\}))$$

be the fuzzy sets (with $J \subseteq I$), then for the extension

$$\{\mu_{\wedge_J \gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\} \qquad \text{for } J \subseteq I$$

with

$$\mu_{\wedge_J \gamma_i}(x) = \varphi(\wedge_{i \in J} E_{\gamma_i} | x) = T_P(\mu_{\gamma_i}(x)) \text{ for any } x \in \mathcal{C}_X$$

of the assessment

$$\{\mu_{\gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\}$$

the following statements hold:

- *the assessment is a coherent conditional plausibility*
- *the assessment is a coherent conditional probability*
- *the assessment is a coherent P-conditional possibility.*

Proof. For the same reasons discussed in the proof of previous Theorem 3.1, we limit the proof to a finite family $\{E_{\gamma_j} | A_x\}_{x \in \mathcal{F}}$ with $\mathcal{F}$ any finite subset of $\mathcal{C}_X$. Since the membership functions $\mu_{\gamma_j}(\cdot)$ is a coherent conditional probability, there is a coherent extension on $\bigwedge_{j \in J} E_{\gamma_j} | A_x$ for any $J \subseteq I$ and $x \in \mathcal{F}$.

For a given $x \in \mathcal{F}_n$, we can define for any $J \subseteq I$

$$f_x(\bigwedge_{i \in J} E_{\gamma_i}) = \Pi_{i \in J} \mu_{\gamma_i}(x)$$

for any set $J \subseteq I$. Any f_x is a probability, so a probability P on the algebra generated by $\{E_{\varphi_i}, A_x : i \in I, x \in \mathcal{F}\}$ can be defined as

$$P(B) = \sum_{A_x \wedge B \neq \emptyset} \frac{1}{n} f_x(B \wedge A_x)$$

(with n the cardinality of $\mathcal{F}$) and it gives rise to a strictly positive probability and it generates a conditional probability that is an extension of $\{\mu_{\gamma_i}\}_I$.

Then, the assignment f is a coherent conditional probability and then a coherent conditional plausibility.

Furthermore, the above assignment f_x, for a given $x \in \mathcal{F}_n$, is a possibilistic distribution and so a possibility Π on the algebra generated by $\{E_{\varphi_i}, A_x : i \in I, x \in \mathcal{F}\}$ can be defined as

$$\Pi(B) = \max_{A_x \wedge B \neq \emptyset} f_x(B \wedge A_x)$$

it corresponds to assign to any x prior possibility equal to 1. □

The above results cannot be extended to any Frank t-norm, in fact by considering the t-norm T_L, the extension to the intersection computed trough T_L can be not a coherent conditional probability, as the following example shows (see [11]).

Example 3.1. Let Let $\mathcal{H} = \{H, H^c\}$ be a partition, and $\mathcal{E} = \{E_i|H\}_{i=1,2,3}$ be a set of conditional events such that $\wedge_{i=1}^{3} E_i^* \wedge H \neq \emptyset$ for any $H \in \mathcal{H}$, so the events in $\mathcal{E}$ are logical independent with respect to $\mathcal{H}$.

Suppose that $P(E_1|H) = P(E_2|H) = 0.6$ and $P(E_3|H) = 0.7$, while $P(E_i|H^c) = 0.5$ for $i = 1, 2, 3$.

It is easy to check that the conditional probability P is coherent. Furthermore it is easy to prove that, from Fréchet-Hoeffdings bounds, the coherent values for P for an event E obtained as finite intersection of E_i are such that:

$$0 \leq P(E_1 \wedge E_2 \wedge E_3|H) \leq 0.6; \quad 0.2 \leq P(E_1 \wedge E_2|H) \leq 0.6;$$

$$0.3 \leq P(E_1 \wedge E_3|H) \leq 0.6 \quad 0.3 \leq P(E_2 \wedge E_3|H) \leq 0.6$$

We could show that the function $f(\wedge_I E_i|H)$ $I \subseteq \{1, 2, 3\}$ taking the minimum coherent values is not coherent: in fact the function

$$f(E_1 \wedge E_2 \wedge E_3|H) = 0, \quad f(E_1 \wedge E_2|H) = 0.2, \quad f(E_1 \wedge E_3|H) = 0.3,$$

$$f(E_1|H) = f(E_2|H) = 0.6, \quad f(E_3|H) = 0.7$$

is not a coherent conditional probability.

However, the t-norm T_L can be used under conditional plausibilities (see [11]).

Theorem 3.3. *Let $\{E^*_{\gamma_i}\}_{i \in I}$ be a finite family of fuzzy sets related to a variable X, with $\{E_{\gamma_i}\}_{i \in I}$ almost logical independent with respect to $\mathcal{C}_X$, let*

$$E^*_{\wedge_{i \in J} \gamma_i} = (E_{\wedge_{i \in J} \gamma_i}, T_L(\{\mu_{\gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\}))$$

be the fuzzy sets (with $J \subseteq I$), then for the extension

$$\{\mu_{\wedge_J \gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\} \qquad \text{for } J \subseteq I$$

with

$$\mu_{\wedge_J \gamma_i}(x) = \varphi(\bigwedge_{i \in J} E_{\gamma_i}|x) = T_L(\mu_{\gamma_i}(x) : i \in J) \text{ for any } x \in \mathcal{C}_X$$

of the assessment

$$\{\mu_{\gamma_i}(x) \text{ with } x \in \mathcal{C}_X \ : \ i \in J\}$$

the assessment is a coherent conditional plausibility.

Proof. Due to the logical independence of the events E_{γ_i} w.r.to X, the assessment $\mathcal{P} = \{Pl(E_{\gamma_i}|x) : i = 1, ..., m\}_{x \in \mathcal{C}_X}$, is coherent with a conditional plausibility; we prove that it can be coherently extended by computing, for any $x \in \mathcal{C}_X$, every intersection trough T_L, that is

$$Pl(\bigwedge_{J \subseteq \{1,\dots,m\}} E_{\gamma_i}|x) = T_L\{Pl(E_{\gamma_i}|x) : i \in J\}, \tag{12}$$

is a coherent conditional plausibility.

Actually we prove the result for $m = 3$ but the basic assignment for any $x \in \mathcal{C}_X$ can be built analogously.

Assume without loss of generality that $Pl(E_{\gamma_i}|x) \le Pl(E_{\gamma_{i+1}}|x)$ for $i = 1, 2$.

Let $m(\wedge_{i=1}^3 E_{\gamma_i}) = T_L(Pl(E_{\gamma_1}|x), ..., Pl(E_{\gamma_3}|x))$,

$m(\vee_{j=1}^3 (E^c_{\gamma_j} \wedge_{i \ne j} E_{\gamma_i})) = T_L(Pl(E_{\gamma_1}|x), Pl(E_{\gamma_2}|x)) - T_L(Pl(E_{\gamma_1}|x), ..., Pl(E_{\gamma_3}|x))$,

$m(E^c_{\gamma_1} \wedge_{j=2}^3 E_{\gamma_j}) = T_L(Pl(E_{\gamma_2}|x), Pl(E_{\gamma_3}|x)) - T_L(Pl(E_{\gamma_1}|x), Pl(E_{\gamma_2}|x))$,

$m(E^c_{\gamma_2} \wedge_{j=1,3} E_{\gamma_j}) = T_L(Pl(E_{\gamma_1}|x), P(E_{\gamma_3}|x)) - T_L(Pl(E_{\gamma_1}|x), Pl(E_{\gamma_2}|x))$,

$m(\vee_{i=1}^3 E_{\gamma_i} \wedge_{j \ne i} E^c_{\gamma_j}) = Pl(E_{\gamma_1}|x) - T_L(Pl(E_{\gamma_1}|x), Pl(E_{\gamma_3}|x))$,

$m(\vee_{i=2}^3 E_{\gamma_i} \wedge_{j \ne i} E^c_{\gamma_j}) = Pl(E_{\gamma_2}|x) - Pl(E_{\gamma_2}|x)$.

Furthermore $m(\wedge_{i=1}^3 E^c_{\gamma_j}) = 1 - Pl(E_{\gamma_3}|x)$.

It is easy to check that the function m taking the above values and zero otherwise is a basic assignment generating the function Pl defined by equation (12), that therefore is coherent.

□

Consider now the problem of complementary which essentially coincides for the three frameworks. Denoting by $(E^*_\gamma)' = E^*_{\neg\gamma} = (E_{\neg\gamma}, \mu_{\neg\gamma})$ the complementary fuzzy set of E^*_γ, due to the logical independence of $\{E_\gamma, E_{\neg\gamma}\}$, with respect to $\mathcal{C}_X$, any value in $[0, 1]$ is coherent for $\mu_{\neg\gamma}(x)$ for any x.

The main remark is related to the fact that the relation $E_{\neg\gamma} \ne E^c_\gamma$ holds. In fact, while $E_\varphi \vee E^c_\gamma = \Omega$, due to the logical independence with respect to $\mathcal{C}_X$ of $\{E_\gamma, E_{\neg\gamma}\}$, we have instead $E_\gamma \vee E_{\neg\gamma} \subseteq \Omega$. Then it is not necessary to require $\mu_{\neg\gamma}(x) = 1$ if $\mu_\gamma(x) < 1$. In particular we can take

$$\mu_{\neg\gamma}(x) = 1 - \mu_\gamma(x). \tag{13}$$

In fact, the above function $\mu_{\neg\gamma}$ is a likelihood function and so a coherent conditional plausibility (as well as a coherent T_P-conditional possibility and a coherent conditional probability).

4. Interval-valued fuzzy sets

Interval-valued fuzzy set (IVF, for short) is a concept introduced by Zadeh in [45], and, independently, by other authors (see [31, 35]).

IVF is defined by a map from the range of a variable X to the set of closed intervals in $[0, 1]$ so that, for every $x \in \mathcal{C}_X$, $\mu(x) \in [a(x), b(x)]$.

One of the interpretation of the interval describing an IVF is based on the idea, introduced in [1], of defining a fuzzy set by a membership function and a non-membership function separately, known as "Intuitionistic Fuzzy Sets".

In [21] motivations for fuzzy sets with two membership functions are reviewed, and the connections between, interval-valued fuzzy sets and possibility theory are studied.

We think that the interpretation of membership of fuzzy sets in therms of coherent conditional plausibility (and in particular probability and possibility), adopted in this paper, can provide an interpretation of interval-valued fuzzy sets which is a "natural" generalization from both a syntactic and a semantic point of view.

Indeed, in this framework, for every $x \in \mathcal{C}_X$, $\mu_\gamma(x) \in [a_\gamma(x), b_\gamma(x)]$ simply means that, when X assumes the value x, the degree of belief of You in E_γ, measured by the function chosen for handling uncertainty, is not less than $a_\gamma(x)$ and not greater than $b_\gamma(x)$.

Definition 4.1. *Let X be a variable with range $\mathcal{C}_X$, γ a property related to X and $a_\gamma(.)$ and $b_\gamma(.)$ two functions from $\mathcal{C}_X$ to $[0,1]$, satisfying condition $(L1)$ and such that $a_\gamma(x) \leq b_\gamma(x)$ for every $x \in \mathcal{C}_X$.*

An **interval-valued fuzzy subset** E_γ^{**} of $\mathcal{C}_X$ is any pair

$$(E_\gamma, [a_\gamma(x), b_\gamma(x)]).$$

An interval-value fuzzy set represents a fuzzy set with "imprecise" membership, that, in our interpretation has a clear meaning: the degree of belief in the event "You claim that X has the property γ, supposing that $X = x$, is a number between $a_\gamma(x)$ and $b_\gamma(x)$, i.e. for every $x \in \mathcal{C}_X$, $\varphi(E_\gamma|x) \in [a_\gamma(x), b_\gamma(x)]$, where φ is the uncertainty measure of reference.

Proposition 4.1. *Let*

$$E_\gamma^{**} = (E_\gamma, [a_\gamma(x), b_\gamma(x)])$$

be a interval-valued fuzzy set, then any assessment $\varphi(E_\gamma|x)$ on $\mathcal{C}_X$ such that, for any $x \in \mathcal{C}_X$

$$a_\gamma(x) \leq \varphi(E_\gamma|x) \leq b_\gamma(x),$$

is a coherent conditional plausibility (or, more specifically, probability or possibility).

Proof. Due to Theorem 2.2 any conditional assessment $P(E|x)$ on a partition $(x \in \mathcal{C}_X)$ is a coherent conditional probability.

Analogously, due to the same Theorem 2.2, any selector $\sigma_{\gamma(x)}$ of $[a_\gamma(x), b_\gamma(x)]$ is a coherent conditional plausibility, a coherent conditional probability and a coherent conditional possibility. □

The above result emphasizes the freedom on assessing initial intervals, independently of the uncertainty framework of reference, in particular it shows that any selector, that is any function from $\mathcal{C}_X$ to [0,1] taking values inside the intervals is a coherent assessments with reference to the aforementioned frameworks.

The next result shows the role of the uncertainty framework in computing intervals related to logical dependent IVF's.

Theorem 4.1. *Let X be a variable, E_γ and E_δ two events almost logical independent with respect to $\mathcal{C}_X$ and $E_\gamma^{**} = (E_\gamma, [a_\gamma(x), b_\gamma(x)]$, $E_\delta^{**} = (E_\delta, [a_\delta(x), b_\delta])$ two IVF. Then the following statements hold:*

- *i) if we refer to conditional plausibility, that is $\mu_\gamma(x) = Pl(E_\gamma|x)$ and $\mu_\delta(x) = Pl(E_\delta|x)$, then one has*

$$E_{\gamma\wedge\delta}^{**} = (E_{\gamma\wedge\delta}, [0, T_M(b_\gamma, b_\delta)], \qquad E_{\gamma\vee\delta}^{**} = (E_{\gamma\vee\delta}, [S_M(a_\gamma, a_\delta), S_L(, b_\gamma, b_\delta]),$$

- *ii) if we refer to conditional probability, that is $\mu_\gamma(x) = P(E_\gamma|x)$ and $\mu_\delta(x) = P(E_\delta|x)$, then one has*

$$E_{\gamma\wedge\delta}^{**} = (E_{\gamma\wedge\delta}, [T_L(a_\gamma, a_\delta), T_M(b_\gamma, b_\delta)], \qquad E_{\gamma\vee\delta}^{**} = (E_{\gamma\vee\delta}, [S_M(a_\gamma, a_\delta), S_L(b_\gamma, b_\delta]),$$

iii) if we refer to conditional possibility, that is $\mu_\gamma(x) = \Pi(E_\gamma|x)$ and $\mu_\delta(x) = \Pi(E_\delta|x)$, then one has

$$E^{**}_{\gamma\wedge\delta} = (E_{\gamma\wedge\delta}, [0, T_M(b_\gamma, b_\delta)], \qquad E^{**}_{\gamma\vee\delta} = (E_{\gamma\vee\delta}, [S_M(a_\gamma, a_\delta), S_M(b_\gamma, b_\delta]).$$

Proof. Taking into account the monotonicity of t-norms and t-conorms, the intervals related to union, intersection and complementation of IVF's will have their minimum by combining the minimal values of the involved intervals and their maximum by combining the maximal values and we obtain the assessments by using Proposition 4.1. and Theorems 3.1 3.2, 3.3.

For plausibilities, to compute $b_{\gamma\wedge\delta}(x)$, we need to apply the same equation to the fuzzy sets $(E_\gamma, b_\gamma(x))$ and $(E_\delta, b_\delta(x))$ and chose the maximum value, that is $T_M(b_\gamma(x), b_\delta(x))$.

For computing $a_{\gamma\vee\delta}(x)$ we need to apply equation (11) to the fuzzy sets $(E_\gamma, a_\gamma(x))$ and $(E_\delta, a_\delta(x))$ and chose the minimum value, that is $S_M(a_\gamma(x), a_\delta(x))$.

To compute $b_{\gamma\wedge\delta}(x)$ we need to apply the same equation to the fuzzy sets $(E_\gamma, b_\gamma(x))$ and $(E_\delta, b_\delta(x))$ and chose the maximum value, that is $\min(b_\gamma(x) + b_\delta(x)) - b_{\gamma\wedge\delta}(x), 1)$. Then, considering again equation (10), we obtain $b_{\gamma\vee\delta}(x) \leq \min(b_\gamma(x) + b_\delta(x), 1)$.

The proof of *ii*) and *iii*) goes along the same line: it is necessary to use equations (6) and (7) for the proof of *ii*), while it is necessary to refer to equations (8) and (9) in order to prove *iii*). □

Now the question is: starting from a set of $\{E_{\varphi_1}, ..., E_{\varphi_n}\}$ of logically independent events with respect to $\mathcal{C}_X$ and the relevant $\mu_i = f(E_{\gamma_i}|x)$ is it possible (i.e. coherent with the measure of reference) to compute all the intersections among the fuzzy sets, by using the same t-norm?

Again the answer is: it depends on the t-norm. If, for instance, we consider the minimum, then the answer is positive in all the considered frameworks of reference:

Theorem 4.2. *Let $\{E^{**}_{\gamma_i}\} = \{(E_{\gamma_i}, [a_{\gamma_i}(x), b_{\gamma_i}(x)] : i \in I\}$ be a finite family of interval-valued fuzzy sets related to a variable X, with E_{γ_i} logically independent with respect to $\mathcal{C}_X$. Let us consider the interval-valued fuzzy set*

$$E^{**}_{\wedge_i\gamma_i} = \big(E_{\wedge_i\gamma_i}, [T_M(a_{\gamma_i}(x) : i \in I), T_M(b_{\gamma_i}(x) : i \in I)]\big)$$

and let $\sigma(x)$ be any selector of $[T_M(a_{\gamma_i}(x) : i \in I), T_M(b_{\gamma_i}(x) : i \in I)]$.
Then the following statements hold:

- *$\sigma(x)$ is a coherent conditional plausibility.*
- *$\sigma(x)$ is a coherent conditional probability,*
- *$\sigma(x)$ is a coherent conditional possibility.*

Proof. The theses follow from Proposition 4 and Theorems 3.1, 3.2 and 3.3. □

The next Theorem 4.3 proves that under a plausibility we can compute, for every $x \in \mathcal{C}_X$, the membership of the intersection of a family of fuzzy sets by using t-norm T_L. This shows that considering coherent conditional plausibility, instead of coherent conditional probability, for measuring the degree of belief of You on the events E_γ, we actually capture more parallelism with the classical theory of fuzzy sets, where the inference is made by using t-norms and t-conorms.

Theorem 4.3. *Let $\{E^{**}_{\gamma_i}\} = \{(E_{\gamma_i}, [a_{\gamma_i}(x), b_{\gamma_i}(x)] : i \in I\}$ be a finite family of interval-valued fuzzy sets related to a variable X, with E_{γ_i} logically independent with respect to $\mathcal{C}_X$. Let us consider the interval-valued fuzzy set*

$$E^{**}_{\wedge_i\gamma_i} = (E_{\wedge_i\gamma_i}, [T_L(a_{\gamma_i}(x) : i \in I), T_L(b_{\gamma_i}(x) : i \in I)]$$

and let $\theta(x)$ be any selector of $[T_L(a_{\gamma_i}(x) : i \in I), T_L(b_{\gamma_i}(x) : i \in I)]$.
Then the following statement holds:

- *$\theta(x)$ is a coherent conditional plausibility.*

Proof. The thesis follows from Proposition 4.1. and Theorem 3.3. □

To make this approach to IVNs more understandable, we present a school example of their use.

Example 4.1. *Let U be an urn containing balls of diameters $\{d_i = 0, i\ cm\}$ $(i = 1, ..., 5)$, whose composition is only partial known, we have only the following information: the balls with diameter d_1 are 1/4, the balls with diameter d_2 or d_3 are 1/4 and those with diameter d_4 or d_5 are 1/2.*

Suppose now to perform an experiment, connected to a bet, consisting in extracting from U a ball, without showing it.

Due to the lack of information on the composition of the urn, the probability of the elements of the algebra spanned by the events D_i = "The drawn ball has diameter d_i" is not unique, depending on two parameters $\theta 1, \theta_2$ with $0 \leq \theta_1 \leq 1/4$ and $0 \leq \theta_2 \leq 1/2$:

$\mathcal{A}$	D_1	D_2	D_3	D_4	D_5	$D_1 \vee D_2$	$D_1 \vee D_3$	$D_1 \vee D_4$	$D_1 \vee D_5$	$D_2 \vee D_3$	...
P	1/4	θ_1	$1/4 - \theta_1$	θ_2	$1/2 - \theta_2$	$1/4 + \theta_1$	$1/2 - \theta_1$	$1/4 + \theta_2$	$3/4 - \theta_2$	1/4	...
$\overline{P}$	1/4	1/4	1/4	1/2	1/2	1/2	1/2	1/2	3/4	1/4	...
$\underline{P}$	1/4	0	0	0	0	1/4	0	1/4	1/4	1/4	...

Since the event D_i form a partition, the upper probability $\overline{P}$ is a plausibility and the lower probability $\underline{P}$ is a belief function [32].

Suppose there is an optimistic decision maker, which in presence of ambiguity, choses, by referring to the upper bound of the class of available (prior) probabilities.

Let us consider now the (Boolean) events

E_s = "You claim that the diameter is small"

and

E_l= "'You claim that the diameter is large"

and let us suppose to require to each person of a a group to express, with a number in $[0, 1]$, his/her degree of belief $\sigma^j_{s_i}$ and $\sigma^j_{l_i}$ $j = 1, ..., m$ on the conditional events $E_s|d_i$ and $E_l|d_i$, with $(i = 1, ..., 5)$.

We recall that the assessment provided by every person j for the two class of conditional events is a coherent conditional probability, possibility and plausibility such as the lower bound $\underline{\sigma}_{l_i} = \min_j \sigma^j_{l_i}$ and $\underline{\sigma}_{s_i} = \min_j \sigma^j_{s_i}$ and the upper bound $\overline{\sigma}_{s_i} = \max_j \sigma^j_{s_i}$ and $\overline{\sigma}_{s_i} = \max_j \sigma^j_{s_i}$ of the class.

Since the uncertainty on the elements of the algebra generated by the diameters is a plausibility, we regard σ_{lj} and σ_{sj} and then $\overline{\sigma}_{s_i}$ $\overline{\sigma}_{l_i}$ and $\underline{\sigma}_{s_i}$ $\underline{\sigma}_{l_i}$ as a coherent conditional plausibility.

So we can consider the two IVF (in the framework of plausibility):

$$(E_s, [\underline{\sigma}_{s_i}, \overline{\sigma}_{s_i}]), \quad (E_l, [\underline{\sigma}_{l_i}, \overline{\sigma}_{l_i}]).$$

5. Fuzzy events

Let us discuss now the concept of fuzzy event, introduced by Zadeh in [44]. In the context of the interpretation of a fuzzy set as a pair, whose elements are a (Boolean) event E_γ and a conditional measure $\sigma(E_\gamma|x)$, coincides exactly with the event

E_γ="You claim that X has property γ".

In presence of an uncertainty measure (probability, possibility and plausibility) on the algebra generated by $\mathcal{C}_X$, the assessment together μ_γ is coherent with respect to the relative measure (see Theorem 2.3) and so coherently extendible to E_γ (Theorem 2.1 for plausibilities, [8] for conditional probabilities and [4, 8] for conditional possibilities).

Finally we consider plausibility of a fuzzy event. If the variable X has finite range, by taking a probability or a T-possibility as "prior", the only coherent value for the probability $P(E_\gamma)$ and possibility $\Pi(E_\gamma)$ are, respectively,

$$P(E_\gamma) = \sum_{x \in \mathcal{C}_X} P(E_\gamma|x)P(x)\,, \tag{14}$$

$$\Pi((E_\gamma) = \max_{x \in \mathcal{C}_X} \Pi(E_\gamma|x)\Pi(x). \tag{15}$$

We note that formula in equation (14) coincides with Zadeh's definition of a probability of a "fuzzy event" given in [43, 44].

The equations (14) and (15) are based on the disintegration formula which hold for both probability and possibility. As discussed in [11] it does not hold for plausibility. In fact, for plausibility just a weak form of disintegration holds, (see inequality in (1)). Then, we need to compute plausibility of an event E_φ by means the Choquet integral (see [3]):

$$Pl(E_\varphi) = \oint \mu_\varphi(x) dPl(x) = \int_0^1 Pl(\mu_\varphi(x) \geq t)dt. \tag{16}$$

Note that

$$Pl(E_\varphi) = \sup_{P \in \mathbf{P}} P(E_\varphi),$$

where $\mathbf{P} = core_{Pl} = \{P : P \leq Pl\}$.

In fact, when a set $\mathbf{P} \subset core_{Pl}$ of probabilities with $\sup \mathbf{P} = Pl$ one has that the extremes of the above interval could be not sharp, that means that the extreme are not obtained by no probability and in this case the vertexes of the closure of the set $\mathbf{P}$ need to be computed.

Let us consider now the IVF $E^{**}_\gamma = (E_\gamma, [a_\gamma(x), b_\gamma(x)]$ and let σ the measure on the algebra spanned by $\mathcal{C}_X$. Then, for the fuzzy event E_γ, we are able to compute the interval of coherent values for its measure of uncertain (plausibility, probability or T_P-possibility), by using in the case of finite $\mathcal{C}_X$ equation (14) to compute probability.

$$P(E_\gamma) \in \big[\sum_{x \in \mathcal{C}_X} a(x)P(x), \sum_{x \in \mathcal{C}_X} b(x)P(x)\big].$$

Under ambiguity, that means that the prior is partially specified we need to refer to no-additive uncertainty measures.

When the prior measure is a possibility by referring to equation (15) when $\mathcal{C}_X$ is finite we obtain

$$\Pi(E_\gamma) \in \big[\max_{x \in \mathcal{C}_X} a(x)\Pi(x), \max_{x \in \mathcal{C}_X} b(x)\Pi(x)\big]$$

When the prior measure is a plausibility by referring to equation (16) we obtain

$$Pl(E_\gamma) \in \big[\int_0^1 Pl(a(x) \geq t)dt, \int_0^1 Pl(b(x) \geq t)dt\big]$$

in the case that the core of plausibilities,

$$core_{Pl} = \{P \,:\, p \leq Pl\}$$

coincides with the coherent extensions of the partially specified prior. When $\mathcal{C}_X$ is finite we could look for the vertexes of the class $\mathbf{P}$ and compute the sharp bounds by means of these set, as aforementioned.

6. Conclusions

To have a consistent framework able to handle together uncertainty and fuzziness, we refer to an interpretation of a fuzzy set consisting in a Boolen event of the kind "You claim that the variable X has the characteristic γ" and a conditional measure regarded as a function of the conditioning event. Then we study which part of the classical theory of fuzzy sets is captured when the uncertainty framework varies in the class of plausibilities. The study puts in evidence the natural develop of the procedure, when the above interpretation is used in situations of partial knowledge on the range of the variables considered. The main results are related to the interpretation of interval-valued fuzzy sets, where the rules of the uncertainty measure of reference naturally permit to manage inference on them.

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References

[1] Atanassov, K., (1986), Intuitionistic fuzzy sets, Fuzzy Sets Syst., 20, pp.87-96.

[2] Bouchon-Meunier, B., Coletti, G., Marsala, C., (2002), Independence and Possibilistic Conditioning, Annals of mathematics and artificial intelligence, 35, pp.107-123.

[3] Choquet, G., (1953), Theory of capacities, Annales de l'Institut Fourier, 5, pp.131-295.

[4] Coletti, G., Petturiti, D., Vantaggi, B., (2017), Fuzzy memberships as likelihood functions in a possibilistic framework, Int. J. Approx. Reason., 88, pp.547-566.

[5] Coletti, G., Scozzafava, R., (2002), Probabilistic Logic in a Coherent Setting. Trends in Logic, 15, Kluwer, Dordrecht, 291p.

[6] Coletti, G., Scozzafava, R., (2004), Conditional probability, fuzzy sets, and possibility: a unifying view, Fuzzy Sets Syst., 144, pp.227-249.

[7] Coletti, G., Scozzafava, R., (2006), Conditional probability and fuzzy information, Comput. Stat. Data Anal., 51, pp.115-132.

[8] Coletti, G., Vantaggi, B., (2009), T-conditional possibilities: coherence and inference, Fuzzy Sets Syst., 160, pp.306-324.

[9] Coletti, G., Vantaggi, B. (2014), Probabilistic Reasoning in a Fuzzy Context. In: Zadeh L., Abbasov A., Yager R., Shahbazova S., Reformat M. (eds), Recent Developments and New Directions in Soft Computing, Stud. Fuzziness Soft. Comput., 317.

[10] Coletti, G., Vantaggi, B., (2017), Knowledge processing in decisions under fuzziness and uncertainty, Proc of Joint 17th World Congress of International Fuzzy Systems Association and 9th International Conference on Soft Computing and Intelligent Systems (IFSA-SCIS), Otsu, pp.1-6.

[11] Coletti, G., Vantaggi, B., (2018), Coherent Conditional Plausibility: A Tool for Handling Fuzziness and Uncertainty Under Partial Information. In: Collan M., Kacprzyk J. (eds), Soft Computing Applications for Group Decision-making and Consensus Modeling, Stud. Fuzziness Soft. Comput., 357, Springer.

[12] Coppi, R., Gil, M.A., Kiers, H.A.L.,(Eds.), (2006), The fuzzy approach to statistical analysis, Comput. Stat. Data Anal., 51, pp.1-452.

[13] de Finetti, B., (1949), Sull'impostazione assiomatica del calcolo delle probabilità, Annali Univ. Trieste, 19, pp.3-55. (Engl. transl.: Ch.5 of Probability, Induction and Statistics, Wiley, London, 1972.)

[14] de Finetti, B., (1970) Teoria della probabilità, Einaudi, Torino, (Engl. transl.(1974): Theory of Probability, 1 and 2, Wiley, Chichester).

[15] Dempster, A.P., (1968), A generalization of Bayesian inference, The Royal Stat. Soc. B, 30(2), pp.205-247.

[16] Denneberg, D., Brüning, M., (2002), Max-min (σ-)additive representation of monotone measures, Stat. Pap., 43(1), pp.23-35.

[17] Dubins, L., (1975), Finitely additive conditional probabilities, conglomerability and disintegrations, Ann. Prob., 3(1), pp.89-99

[18] Dubois, D., Moral, S., Prade, H., (1997), A semantics for possibility theory based on likelihoods, J. Math. Anal. Appl., 205, pp.359-380.

[19] Dubois, D., Prade, H., (1986), New results about properties and semantics of fuzzy set-theoretic operators, in: P.P. Wang and S.K. Chang, Eds., Fuzzy Sets., Plenum Press, New York, pp.59-75.

[20] Dubois, D., Prade, H., (2005), Interval-valued fuzzy sets, possibility theory and imprecise probability, In Proceedings of International Conference in Fuzzy Logic and Technology, pp.314-319.

[21] Dubois, D., and Prade, H., (1997), The three semantics of fuzzy sets, Fuzzy Sets Syst., 90, pp.141-150.

[22] Dubois, D., (2006), Possibility theory and statistical reasoning, Comput. Stat. Data Anal., 51, pp.47-69.

[23] Frank, M.J., (1979), On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, Aequ. Math., 19, pp.194-226.

[24] Hamachar, H., (1978), Uber logische verknunpfungenn unssharfer Aussagen und deren Zugenhorige Bewertungsfunktione, in: Trappl, Klir, Riccardi(Eds.), Progress in Cybernetics and systems research, 3, Hemisphere, Washington DC, pp.276-288.

[25] Jaffray, J.Y., (1992), Bayesian Updating and Belief Functions, IEEE Trans. Syst. Man Cybern., 22, pp.1144-1152.

[26] Krauss, P.H., (1968), Representation of conditional probability measures on Boolean algebras, Acta Mathematica Academiae Scientiarum Hungaricae, 19(3-4), pp.229-241.

[27] Lancaster, J.L., et al., (2000), Automated talairach atlas labels for functional brain mapping, Human Brain Mapping, 10, pp.120-131.

[28] Loginov, V.I., (1966), Probability treatment of Zadeh membership functions and their use in patter recognition, Engineering Cybernetics, pp.68-69.

[29] Fruhwirth-Schnatter, S., (1993), On fuzzy bayesian inference, Fuzzy Sets Syst., 60, pp.41-58.

[30] Gil, M.A., Lopez-Diaz, M., Ralescu, D.A., (2006), Fuzzy sets and probability/Statistics Theories, Fuzzy Sets Syst., 157, pp.2545-2682.

[31] Grattan-Guiness, I., (1975), Fuzzy membership mapped onto interval and many-valued quantities, Z. Math. Logik, Grundladen Math., 22, pp.149-160.

[32] Fagin R., Halpern, J.Y., (1991) Uncertainty, belief, and probability, Computational Intelligence, 7(3), pp.160-173.

[33] Lubiano, M. A., de Saa, S.R. Montenegro, M., Sinova, B., Gil, M. A., (2016), Descriptive analysis of responses to items in questionnaires. Why not using a fuzzy rating scale?, Inf., Sci., 360, pp.131-148.

[34] Neumaier, A., (2004), Clouds, fuzzy sets and probability intervals, Reliable Computing, 10, pp.249-272.

[35] Jahn, K.U., (1975), Intervall-wertige Mengen, Math.Nach., 68, pp.115-132.

[36] Scozzafava, R., Vantaggi, B., (2009), Fuzzy inclusion and similarity through coherent conditional probability, Fuzzy Sets Syst., 160, pp.292-305.

[37] Shafer, G., (1976), A Mathematical Theory of Evidence, Princeton University Press, 314p.

[38] Singpurwalla, N.D., Booker, J.M., (2004), Membership functions and probability measures of fuzzy sets (with discussion), Journal of America Statist. Association, 99, pp.867-889.

[39] Viertl, R., (1987), Is it necessary to develop a fuzzy Bayesian inference? in: R. Viertl (Ed.), Probability and Bayesian Statistics, pp.471-475.

[40] Viertl, R., (1996), Statistical Methods for Non-Precise Data, CRC Press, Boca Raton, 191p.

[41] Yager, R.R., (1980), On a general class of fuzzy connectives, Fuzzy Sets Syst., 4, pp.235-242.

[42] Walley, P., (1991), Statistical reasoning with Imprecise Probabilities, London: Chapman and Hall, 720p.

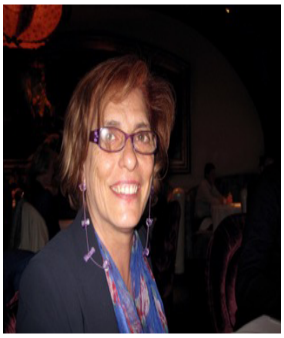
[43] Zadeh, L.A., (1965), Fuzzy sets, Inf. Control., 8, pp.338-353.

[44] Zadeh, L.A., (1968), Probability measures of fuzzy events, J. Math. Anal. Appl., 23, pp.421-427.

[45] Zadeh, L.A., (1975), The concept of a linguistic variable and its application to approximate reasoning- I, Inf. Sci., 8, pp.199-249.

[46] Zadeh, L.A., (2002), Toward a perception-based theory of probabilistic reasoning with imprecise probabilities, J. Stat. Plan. Inference., 105, pp.233-264.

[47] Zheng, J., Tang, Y., (2005), On generalization of the Naive Bayes to fuzzy sets and the design of fuzzy Naive Bayes classifier, LNCS 3562, pp.281-290.



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